theorem and to obtain more precise results for special classes of generating functions. An advantage of this approach is that, since the theory of the associated functional equations is not used, the Appell expansion becomes available as a tool for studying the functional equations. Furthermore, the method works equally well for a large class of the more general polynomial sets defined by

$$A(t)e^{zt} = \sum_{n=0}^{\infty} \{B(t)\}^n p_n(z),$$

where $B(0) = 0$, $B'(0) = 1$ (Sheffer's "sets of type 0") and the results so obtained could also be applied to functional equations.

To carry out the discussion outlined here requires more information about the reciprocal of an entire function than seems to be available in the literature. The details will be given elsewhere.

[1] Sheffer, I. M., "Some Applications of Certain Polynomial Classes," *Bull. Am. Math. Soc.*, **47**, 885–898 (1941); further references are given there.

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# *NOTES ON INTEGRATION, III*

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As part of our theory of general integration begun in earlier notes,[1] we shall now establish general forms of the Fubini theorem and its extensions. Since the Fubini theorem deals with multiple and iterated integration, our notation must be modified so that at least three general integrations, attached to as many different domains, can be handled simultaneously without confusion. For our present purposes it suffices to indicate clearly the particular domain to which is attached each mathematical object under consideration. Thus the different families of functions which have to be considered on a given domain $X$ will be denoted as $\mathfrak{E}(X)$, $\mathfrak{G}(X)$, $\mathfrak{L}(X)$, and so on. Operations on such functions will be denoted, in a slightly different fashion, as $E_x$, $N_x$, $L_x$, and so on; and the results of applying such operations to a particular function $f$ will be denoted as $E_xf(x)$, $N_xf(x)$, $L_xf(x)$, and so on.[2] Furthermore it will be convenient to follow the common practice in shortening the precise phrase "the function $f$ whose value at $x$ is $f(x)$" to the handier phrase "the function $f(x)$."

Let $Z = X \times Y$ be the Cartesian product of $X$ and $Y$—that is, the totality of pairs $z = (x, y)$ where $x \in X$ and $y \in Y$. Let $E_x$ and $E_y$ be elementary integrals defined for the respective families $\mathfrak{E}(X)$ and $\mathfrak{E}(Y)$ of elementary functions. We then designate by $\mathfrak{E}(X) * \mathfrak{E}(Y)$ the totality of real functions $f(z) = f(x, y)$ with the following properties: for fixed $x$, the function $f(x, y)$ is in $\mathfrak{E}(Y)$; and the integral $E_y f(x, y)$ is a function in $\mathfrak{E}(X)$. This family is obviously linear but is not guaranteed to contain $|f|$ whenever it contains $f$; it fails in this respect to conform to the requirements imposed upon a family of elementary functions. Further, we designate by $E_x * E_y$ the operation which takes any function $f$ with the above properties into the real number $E_x E_y f(x, y)$. This is a positive linear operation which even satisfies the condition I (2) under the hypothesis that $|f|$ and $|f_n|$ are in $\mathfrak{E}(X) * \mathfrak{E}(Y)$: indeed, if $|f(x, y)| \leqq \sum_{n=1}^{\infty} |f_n(x, y)|$, application of I (2) to $E_y$ yields $E_y|f(x, y)| \leqq \sum_{n=1}^{\infty} E_y|f_n(x, y)|$ for each $x$ and then application of I (2) to $E_x$ yields $E_x E_y|f(x, y)| \leqq \sum_{n=1}^{\infty} E_x E_y|f_n(x, y)|$. If $\mathfrak{E}(Z)$ is a linear subfamily of $\mathfrak{E}(X) * \mathfrak{E}(Y)$ which contains $|f|$ together with $f$ and if $E_z$ is the contraction of $E_x * E_y$ to $\mathfrak{E}(Z)$, we therefore see that $\mathfrak{E}(Z)$ and $E_z$ satisfy I (1) and I (2); and hence that $E_z$ can be regarded as an elementary integral and $\mathfrak{E}(Z)$ as the family of elementary functions over which it is defined. There are many important examples where $\mathfrak{E}(Z)$ can be specified so as to contain all the functions $h(z) = h(x, y) = f(x)g(y)$ where $f \epsilon \mathfrak{E}(X)$ and $g \epsilon \mathfrak{E}(Y)$, such functions $h$ obviously being members of $\mathfrak{E}(X) * \mathfrak{E}(Y)$. Frequently $E_z$ is given directly without reference to $E_x$ and $E_y$, and has to be identified as a contraction of $E_x * E_y$—in other words, the relation $E_z \subset E_x * E_y$ has to be proved as a theorem. Since illustrations of these remarks are well known we shall not go into greater detail here. We must, however, call particular attention to the fact that in general we will have $\mathfrak{E}(X) * \mathfrak{E}(Y) \neq \mathfrak{E}(Y) * \mathfrak{E}(X)$ and $E_x * E_y \neq E_y * E_x$. This lack of symmetry may well extend so far that for some functions $f$ we have $E_x E_y f(x, y) \neq E_y E_x f(x, y)$; but, of course, the functions $h(z) = h(x, y) = f(x)g(y)$ where $f \epsilon \mathfrak{E}(X)$ and $g \epsilon \mathfrak{E}(Y)$ are not among them. On the other hand there are many familiar and important cases where $\mathfrak{E}(Z)$ can be specified so that $E_x * E_y$ and $E_y * E_x$ have identical contractions to $\mathfrak{E}(Z)$. In such a case we have $E_z f(z) = E_x E_y f(x, y) = E_y E_x f(x, y)$ and the relation of $E_z$ to $E_x$ and $E_y$ involves the latter in a symmetric manner.

Turning now to the general integrations $L_x$ and $L_y$ associated with $E_x$ and $E_y$, respectively, we shall introduce an operation $L_x * L_y$ analogous to the operation $E_x * E_y$ of the preceding paragraph. First we designate by $\mathfrak{L}(X) * \mathfrak{L}(Y)$ the family of all extended-real functions $f(z) = f(x, y)$ with the following properties: for each $x$ outside a fixed null subset $X_0$ of $X$ the function $f(x, y)$ is in $\mathfrak{L}(Y)$; and the integral $L_y f(x, y)$ is a function defined outside $X_0$ and coinciding there with a function $g$ in $\mathfrak{L}(X)$. We then define $L_x * L_y$ as the operation which takes such a function $f$ into the real number $L_x g(x)$ where $g(x) = L_y f(x, y)$ for $x$ outside $X_0$ and $g \in \mathfrak{L}(X)$, observing that this number is unaffected by the ambiguity in the determination of $g$. As we shall see below, the generalized Fubini theorem is conveniently expressed in terms of the operation $L_x * L_y$.

As a preliminary to the statement and proof of the generalized Fubini theorem, we first make the following observation:

(1) *if the elementary integrations $E_x$, $E_y$, and $E_z$ satisfy the relation $E_z \subset E_x * E_y$, then the corresponding operations $N_x$, $N_y$, $N_z$ are such that $N_z f(z) \geqq N_x N_y f(x, y)$ for every $f$ in $\mathfrak{G}(Z)$.*

The proof is simple. Since we have nothing to prove unless $N_z f(z) < +\infty$, we assume the latter relation. We can then choose $f_n$ in $\mathfrak{E}(Z)$ so that $|f| \leqq \sum_{n=1}^{\infty} |f_n|$ and $\sum_{n=1}^{\infty} E_x E_y |f_n(x, y)| = \sum_{n=1}^{\infty} E_z |f_n(z)| \leqq N_z f(z) + \epsilon$ for any given $\epsilon > 0$. On the other hand we have $N_y f(x, y) \leqq \sum_{n=1}^{\infty} N_y f_n(x, y) = \sum_{n=1}^{\infty} E_y |f_n(x, y)|$ by I (7) and I (9); and hence $N_x N_y f(x, y) \leqq \sum_{n=1}^{\infty} N_x E_y |f_n(x, y)| = \sum_{n=1}^{\infty} E_x E_y |f_n(x, y)| \leqq N_z f(z) + \epsilon$ by I (7), I (9), and the above. Since $\epsilon > 0$ is arbitrary, the theorem is established.

We now state the first part of the generalized theorem of Fubini:

(2) (Fubini) *if the elementary integrations $E_x$, $E_y$, and $E_z$ satisfy the relation $E_z \subset E_x * E_y$, then the corresponding general integrations satisfy the relation $L_z \subset L_x * L_y$—in other words the general integral $L_z f(z)$ can be evaluated as the iterated integral $L_x L_y f(x, y)$ in the sense made precise above.*

With the help of (1) the proof offers little difficulty. If $f$ is in $\mathfrak{L}(Z)$ we can find $f_n$ in $\mathfrak{E}(Z)$ so that $N_z(f(z) - f_n(z)) \leqq 2^{-n}$. The positive-term series $\sum_{n=1}^{\infty} N_y(f(x, y) - f_n(x, y))$ has sum $h(x)$ in $\mathfrak{G}(X)$; and the relation $N_x h(x) \leqq \sum_{n=1}^{\infty} N_x N_y(f(x, y) - f_n(x, y)) \leqq \sum_{n=1}^{\infty} N_z(f(z) - f_n(z)) \leqq \sum_{n=1}^{\infty} 2^{-n} = 1$ shows that $h \in \mathfrak{F}(X)$ and hence that $h$ is finite except on a null set $X_0$. If $x$ is outside $X_0$ we therefore have $\lim_{n\to\infty} N_y(f(x, y) - f_n(x, y)) = 0$ and hence $f(x, y) \in \mathfrak{L}(Y)$. We let $g(x)$ be any function in $\mathfrak{G}(X)$ which is equal to $L_y f(x, y)$ outside $X_0$. Since $|g(x) - E_y f_n(x, y)| = |L_y f(x, y) - E_y f_n(x, y)| = |L_y(f(x, y) - f_n(x, y))| \leqq L_y |f(x, y) - f_n(x, y)| = N_y(f(x, y) - f_n(x, y))$ almost everywhere, we see that $N_x(g(x) - E_y f_n(x, y)) \leqq N_x N_y(f(x, y) - f_n(x, y)) \leqq N_z(f(z) - f_n(z)) \leqq 2^{-n}$. Hence $g(x)$ is in $\mathfrak{L}(X)$ and $L_x g(x) = \lim_{n\to\infty} E_x E_y f_n(x, y) = \lim_{n\to\infty} E_z f_n(z) = L_z f(z)$ in accordance with the definitions of $L_x$ and $L_z$. The remainder of the generalized theorem of Fubini is the following partial converse of (2):

(3) (Fubini) *if the elementary integrations* $E_x$, $E_y$, *and* $E_z$ *satisfy the relation* $E_z \subset E_x * E_y$, *if* $f \in \mathfrak{M}(Z)$, *and if there exist functions* $f_n$ *in* $\mathfrak{F}(Z)$ *such that* $|f| \leqq \sum_{n=1}^{\infty} |f_n|$, *then* $|f| \in \mathfrak{L}(X) * \mathfrak{L}(Y)$ *implies* $f \in \mathfrak{L}(Z)$.

In accordance with II (14) we may suppose without loss of generality that $f_n \in \mathfrak{L}(Z)$. The function $g_n = \min(|f|, |f_1| + \ldots + |f_n|)$ is also in $\mathfrak{L}(Z)$ in accordance with II (6), II (7). Since $0 \leqq g_n \leqq |f|$ we see that $N_y g_n(x, y) \leqq N_y f(x, y)$ and $N_x N_y g_n(x, y) \leqq N_x N_y f(x, y)$. The fact that $|f|$ is in $\mathfrak{L}(X) * \mathfrak{L}(Y)$ shows that $N_y f(x, y) = L_y |f(x, y)|$ for almost all $x$, and also that $N_y f(x, y)$ differs only on a null set from an integrable function $g(x)$ and is therefore integrable itself. Thus $N_x N_y f(x, y) = L_x g(x) < +\infty$. On the other hand the inequalities $g_n \leqq g_{n+1}$ and $|f| \leqq \sum_{n=1}^{\infty} |f_n|$ show that $\{g_n\}$ is a monotonely increasing sequence which converges to $|f|$. Hence $N_z f(z) = \lim_{n \to \infty} L_z g_n(z)$. Application of (2) to $g_n \in \mathfrak{L}(Z)$ yields $L_z g_n(z) = N_x N_y g_n(x, y) \leqq N_x N_y f(x, y)$. Thus $N_z f(z) \leqq N_x N_y f(x, y) < +\infty$ so that $f \in \mathfrak{F}(Z)$. Since it was assumed that $f \in \mathfrak{M}(Z)$, we conclude by II (11) that $f \in \mathfrak{L}(Z)$. It is well known that the hypotheses of (3) cannot be weakened in any essential respect. Membership in $\mathfrak{L}(X) * \mathfrak{L}(Y)$ does not imply membership in $\mathfrak{M}(Z)$, so that some hypothesis concerning the measurability of $f$ is needed in order to guarantee that $f \in \mathfrak{L}(Z)$. The need for the condition that $|f| \leqq \sum_{n=1}^{\infty} |f_n|$ for appropriate $f_n$ in $\mathfrak{F}(Z)$ is illustrated by a simple example of Saks.[3] This condition is automatically satisfied in many of the common instances of our general theory. In particular if $1 \in \mathfrak{L}(Z)$ we can always take $f_n = 1$.

Our version of the Fubini theorem can be applied directly to a situation in the theory of locally compact topological groups.[4] Let $Z$ be such a group, $Y$ one of its closed subgroups and $X$ the homogeneous space of left cosets of $Y$. Selecting from each coset $x$ a fixed element $z_x$ we see that the equation $z = z_x y$ defines a one-to-one correspondence between $Z$ and $X \times Y$, which can therefore be identified as abstract sets during the remainder of the discussion. We let $\mathfrak{C}(X)$, $\mathfrak{C}(Y)$, $\mathfrak{C}(Z)$ be the families of continuous real functions with compact nuclei on the respective spaces $X$, $Y$, $Z$. Three elementary integrations $E_x$, $E_y$, $E_z$ defined over $\mathfrak{C}(X)$, $\mathfrak{C}(Y)$, $\mathfrak{C}(Z)$, respectively, will be said to form an admissible triple if $E_y$ is left-invariant, $E_x$ and $E_z$ are relatively left-invariant and $E_z f(z) = E_x E_y f(z'y)$ for all $f$ in $\mathfrak{C}(Z)$. It is implicit in this definition that the integral $E_y f(z'y)$ is constant on each coset $x$ and can therefore be considered as a function on $X$ which is, in fact, a member of $\mathfrak{C}(X)$. We observe now that $E_x$, $E_y$, $E_z$ constitute an admissible triple if and only if, in addition to enjoying the required invariance properties, they satisfy the relation $E_z \subset E_x * E_y$. To verify this we need only note that for any $f$ in $\mathfrak{E}(Z)$ we have $f(z) = f(z_xy) = f(x, y)$ and hence $E_yf(x, y) = E_yf(z_xy) = E_yf(z'y)$ for all $z'$ in the coset $x$: for we then have $E_yf(x, y) \in \mathfrak{E}(X)$ and the conditions $E_zf(z) = E_xE_yf(z'y)$ and $E_z \subset E_x * E_y$ are therefore equivalent. The group-theoretic conditions for the existence of admissible triples are discussed by A. Weil.[4] Here we direct attention to the fact that when $\{E_x, E_y, E_z\}$ is an admissible triple, *the associated general integrations* $L_x$, $L_y$, $L_z$ (which enjoy corresponding invariance properties) *must satisfy the relation* $L_z \subset L_x * L_y$ *in accordance with* (2) *above.*[5]

We turn finally to an extension of the Fubini theorem, due originally to Jessen in a particular case.[6] With each element $\lambda$ of a fixed infinite class $\Lambda$ let there be associated a non-void set $X_\lambda$. We denote by $X_A$ the Cartesian product of those $X_\lambda$ with $\lambda \in A$. For each non-void *finite* part B of $\Lambda$ let there be given an elementary integration $E_{x_B}$ defined for a family $\mathfrak{E}(X_B)$ of elementary functions on $X_B$. Let it be assumed that the following conditions hold:

(4) *the constant function everywhere equal to* 1 *on* $X_B$ *is in* $\mathfrak{E}(X_B)$ *and its elementary integral is* 1;

(5) *if* $\Gamma$ *and* $\Delta$ *constitute a partition of* B *and if* $g \in \mathfrak{E}(X_\Gamma)$, *then the function f defined by* $f(x_B) = f(x_\Gamma, x_\Delta) = g(x_\Gamma)$ *is in* $\mathfrak{E}(X_B)$;

(6) *if* $\Gamma$ *and* $\Delta$ *constitute a partition of* B, *then* $E_{x_B} \subset E_{x_\Gamma} * E_{x_\Delta}$.

For an arbitrary infinite part A of $\Lambda$ we can now define[7] $\mathfrak{E}(X_A)$ and $E_{x_A}$ satisfying I (1): we take $\mathfrak{E}(X_A)$ to be the family of those functions $f$ on $X_A$ such that for some partition of A into a finite set B and its complement $\Gamma$ and for some $g$ in $\mathfrak{E}(X_B)$ the relation $f(x_A) = f(x_B, x_\Gamma) = g(x_B)$ is valid; and for each such $f$ we put $E_{x_A}f(x_A) = E_{x_B}f(x_B, x_\Gamma) = E_{x_B}g(x_B)$. We assume finally that I (2) holds for $\mathfrak{E}(X_\Lambda)$ and $E_{x_\Lambda}$. It is then evident that I (2) must also hold for $\mathfrak{E}(X_A)$ and $E_{x_A}$, whatever the infinite set $A \subset \Lambda$. As an instance where all our assumptions can easily be verified, we cite one equivalent to that given by Jessen:[6] we take $X_\lambda$ to be the unit interval, $0 \leqq x_\lambda \leqq 1$; $\mathfrak{E}(X_B)$ to be the family of all continuous real functions on the hypercube $X_B$; and $E_{x_B}$ to be the Riemann integral. In the general case a rather simple analysis, which we shall not repeat, shows that

(7) *in* (4), (5) *and* (6) *the finite set* B *can be replaced by an arbitrary infinite set* $A \subset \Lambda$.

We now direct attention to a remarkable property of the general integration associated with $E_{x_\Lambda}$, namely:

(8) *if* $f \in \mathfrak{L}_p(X_\Lambda)$ *then there exist a partition of* $\Lambda$ *into sets* $\Gamma$ *and* $\Delta$, *where* $\Gamma$ *is countable, and a function* g *in* $\mathfrak{L}_p(X_\Gamma)$ *such that* $f(x_\Lambda) = f(x_\Gamma, x_\Delta) = g(x_\Gamma)$ *for almost all* $x_\Lambda$.

Because of the mapping of $\mathfrak{L}_p(X_A)$ onto $\mathfrak{L}(X_A)$ discussed in II, it suffices to treat the case $p = 1$. There we determine functions $f_n$ in $\mathfrak{C}(X_A)$ such that $N(f - f_n) \leqq 2^{-n}$. As in the proof of I (10), we find that $f$ and $\limsup_{n\to\infty} f_n = h$ differ only on a null set. In view of the definition of $\mathfrak{C}(X_A)$ there exists a countable set $\Gamma$ such that for every $n$ the function $f_n(x_A) = f_n(x_\Gamma, x_\Delta)$ is constant with respect to $x_\Delta$. Hence $h(x_A) = h(x_\Gamma, x_\Delta)$ has the same property. The function $g(x_\Gamma) = h(x_\Gamma, x_\Delta)$ is then in $\mathfrak{L}(X_\Gamma)$, since (6) and (7) above permit the application of (2). Jessen's most interesting results concern the case where A is countable; but (8) shows that in handling a finite or countably infinite family of functions in $\mathfrak{L}_p(X_A)$ there is no loss of generality in restricting attention to this case. We suppose therefore that A is the class of positive integers, and establish the following result:

(9) (Jessen) *if* $f \in \mathfrak{L}_p(X_A)$, $B = \{\alpha;\ \alpha \leqq n\}$, *and* $\Gamma = \{\alpha;\ \alpha > n\}$, *then there exist functions* $g_n$, $h_n$ *in* $\mathfrak{L}_p(X_A)$ *defined for almost all* $X_A$ *by the relations* $g_n(x_A) = g_n(x_B, x_\Gamma) = L_{x_B} f(x_B, x_\Gamma)$, $h_n(x_A) = h_n(x_B, x_\Gamma) = L_{x_\Gamma} f(x_B, x_\Gamma)$; *and in* $\mathfrak{L}_p(X_A)$ *the convergence relations* $g_n \to L_{x_A} f(x_A)$, $h_n \to f$ *are valid.*

Only $h_n$ will be discussed in detail, as $g_n$ can be treated in much the same way. Since $1 \in \mathfrak{L}(X_\Delta)$ for any $\Delta \subset A$, we see that $\mathfrak{L}_p(X_\Delta) \subset \mathfrak{L}(X_\Delta)$ and that all the techniques developed in II are available to us here. The Fubini theorem is also available to us. Thus we can use this theorem to infer first that $h_n$ exists as a member of $\mathfrak{L}(X_A)$. If $f$ is in $\mathfrak{L}_p(X_A)$, the inequality $N_{x_A}|h_n(x_A)|^p = N_{x_B}|L_{x_\Gamma} f(x_B, x_\Gamma)|^p \leqq N_{x_B} L_{x_\Gamma}|f(x_B, x_\Gamma)|^p = L_{x_B} L_{x_\Gamma}|f(x_B, x_\Gamma)|^p = L_{x_A}|f(x_A)|^p = N_{x_A}|f(x_A)|^p < +\infty$ shows that $h_n$ is also in $\mathfrak{L}_p(X_A)$. If $f$ is in $\mathfrak{L}_p(X_A)$ we can find for any $\epsilon > 0$ a function $\bar{f}$ in $\mathfrak{L}_p(X_A)$ such that $\bar{f}(x_A) = \bar{f}(x_B, x_\Gamma)$ is constant with respect to $x_\Gamma$ for some choice of $B = \{\alpha;\ \alpha \leqq m\}$ while $N_p(f - \bar{f}) \leqq \frac{1}{2}\epsilon$. Using the notations of II, we choose $\bar{f}$ so that $\bar{g} = \Phi(\bar{f}) \in \mathfrak{C}(X_A) \subset \mathfrak{L}(X_A)$ and $N(\Phi(f) - \bar{g}) \leqq \delta$. Since $B = \{\alpha;\ \alpha \leqq m\}$ can be chosen so that $\bar{g}(x_A) = \bar{g}(x_B, x_\Gamma)$ is constant with respect to $x_\Gamma$, we see that $\bar{f} = \Psi(\bar{g})$ has a like property and belongs to $\mathfrak{L}_p(X_A)$. If $\delta$ has been taken sufficiently small, it is clear that $N_p(f - \bar{f}) \leqq \frac{1}{2}\epsilon$ by virtue of the continuity of $\Psi$. Now if $n \geqq m$, $m$ being the integer just determined in our choice of $\bar{f}$, we see that $\bar{h}_n(x_A) = L_{x_\Gamma}\bar{f}(x_B, x_\Gamma) = \bar{f}(x_B, x_\Gamma) = \bar{f}(x_A)$ because $\bar{f}(x_B, x_\Gamma)$ is constant with respect to $x_\Gamma$, $\Gamma = \{\alpha;\ \alpha > n\}$. Hence we have $N_p(f - h_n) = N_p(f - \bar{f} + \bar{h}_n - h_n) \leqq N_p(f - \bar{f}) + N_p(\bar{h}_n - h_n)$ for $n \geqq m$, by Minkowski's inequality. By our choice of $\bar{f}$ we have $N_p(f - \bar{f}) \leqq \frac{1}{2}\epsilon$ and $N_p(\bar{h}_n - h_n) = (L_{x_A}|\bar{h}_n(x_A) - h(x_A)|^p)^{1/p} = (L_{x_B} L_{x_\Gamma}|L_{x_\Gamma}(\bar{f}(x_B, x_\Gamma) - f(x_B, x_\Gamma))|^p)^{1/p} = (L_{x_B}|L_{x_\Gamma}(\bar{f}(x_B, x_\Gamma) - f(x_B, x_\Gamma)|^p)^{1/p} \leqq (L_{x_B} L_{x_\Gamma}|\bar{f}(x_B, x_\Gamma) - f(x_B, x_\Gamma)|^p)^{1/p} = (L_{x_A}|\bar{f}(x_A) - f(x_A)|^p)^{1/p} = N_p(f - \bar{f}) \leqq \frac{1}{2}\epsilon$. Hence $N_p(f - h_n) \leqq \epsilon$ for $n \geqq m$, as we wished to show. A further result of Jessen will complete our discussion, namely:

(10) (Jessen) *the relations* $\lim_{n\to\infty} g_n(x_A) = L_{x_A} f(x_A)$, $\lim_{n\to\infty} h_n(x_A) = f(x_A)$ *hold almost everywhere in the pointwise sense.*

We consider only $h_n$, modeling our treatment on that already given by Jessen[6] for $g_n$. Since $\mathfrak{L}_p(X_A) \subset \mathfrak{L}(X_A)$ we may suppose that $p = 1$ in the present instance. The sequence $\{h_n\}$, being convergent to $f$ in $\mathfrak{L}(X_A)$ has a subsequence which converges almost everywhere to $f$ in the pointwise sense. Consequently $h = \limsup_{n\to\infty} h_n \geqq f$ almost everywhere. By a method which will be sketched below we show that at almost every point of the set $X^\lambda = \{x_A;\ h(x_A) > \lambda\}$ we must have $f(x_A) \geqq \lambda$. It then follows that $\limsup_{n\to\infty} h_n(x_A) = f(x_A)$ almost everywhere. Replacing $f$ by $-f$, we have to replace $h_n$ by $-h_n$. We therefore have $\liminf_{n\to\infty} h_n(x_A) = -\limsup_{n\to\infty} (-h_n(x_A)) = -(-f(x_A)) = f(x_A)$, a relation which completes the proof of the theorem. Reverting now to the detailed study of $X^\lambda$, we let $f^\lambda_{np} \in \mathfrak{L}(X_A)$ be the characteristic function of the set $\{x_A;\ h_{n+p}(x_A) > \lambda,\ h_k(x_A) \leqq \lambda$ *for* $n \leqq k \leqq n + p - 1\}$, $n = 1, 2, 3, \ldots$ and $p = 0, 1, 2, \ldots$. The characteristic function of $X^\lambda$ is then expressible as $f^\lambda = \lim_{n\to\infty} \sum_{p=0}^{\infty} f^\lambda_{np} \in \mathfrak{L}(X_A)$. Since $h_k$ is constant with respect to $x_l$ for $l \geqq k + 1$ we see that $h_{n+p}$ and $f_{np}$ are both constant with respect to $x_l$ for $l \geqq n + p + 1$. Let $g$ be an arbitrary function in $\mathfrak{L}(X_A)$ which is constant with respect to $x_l$ for $l \geqq m + 1$ and which satisfies the inequalities $0 \leqq g \leqq 1$. Taking $n \geqq m$ we note that $f^\lambda_{np}g$ is constant with respect to $x_l$ for $l \geqq n + p + 1$ and hence can be multiplied into both members of the equation $h_{n+p}(x_A) = L_{x_\Gamma} f(x_B, x_\Gamma)$ to yield $h_{n+p}(x_A) f^\lambda_{np}(x_A) g(x_A) = L_{x_\Gamma}[f(x_B, x_\Gamma) f^\lambda_{np}(x_B, x_\Gamma) g(x_B, x_\Gamma)]$. Applying $L = L_{x_A}$ to both members of the latter equation we obtain $L(h_{n+p} f^\lambda_{np} g) = L(f f^\lambda_{np} g)$. Since $h_{n+p} > \lambda$ on the set where $f^\lambda_{np} = 1$ it follows that $\lambda L(f^\lambda_{np} g) \leqq L(f f^\lambda_{np} g)$ and hence that $\lambda L(f^\lambda g) \leqq L(f f^\lambda g)$. It is not difficult to determine a sequence of functions $g$ of the kind admitted here which converges boundedly to the characteristic function $g^{\lambda-\epsilon}$ of the set $\{x_A;\ f(x_A) \leqq \lambda - \epsilon\}$, exception being made as usual for points of a null set. Passage to the limit in the above inequality therefore yields $\lambda L(f^\lambda g^{\lambda-\epsilon}) \leqq L(f f^\lambda g^{\lambda-\epsilon}) \leqq (\lambda - \epsilon) L(f^\lambda g^{\lambda-\epsilon})$. Hence $\epsilon > 0$ implies $L(f^\lambda g^{\lambda-\epsilon}) = 0$; in other words the part of $X^\lambda$ where $f(x_A) \leqq \lambda - \epsilon$ is a null set. Thus we must have $f(x_A) \geqq \lambda$ almost everywhere on $X_A$, as we claimed above.

[1] Stone, M. H., "Notes on Integration, I," these PROCEEDINGS, **34**, 336–342 (1948); "Notes on Integration, II," *Ibid.* 447–455 (1948); cited here as I and II, respectively.

[2] The symbol $x$ in these expressions denotes a bound variable.

[3] Saks, S., *Theory of the Integral*, 2nd revised ed., Warszawa-Lwow, 1937, pp. 87–88.

[4] Weil, A., *L'Intégration dans les Groupes Topologiques et ses Applications*, Paris, 1938, pp. 30–45, especially 42–45.

[5] While this result resembles one established by Ambrose, W., "Direct Sum Theorem for Haar Measures," *Trans. Am. Math. Soc.*, **61**, 122–127 (1947), it is actually identical with the latter only in the case where $Z$ is separable. The reason for the distinction which must be made in the non-separable case is indicated in the fourth footnote of II.

[6] Jessen, B., "The Theory of Integration in a Space of an Infinite Number of Dimensions," *Acta Mathematica*, **63**, 249–323 (1934), especially 272–280.

[7] $\mathfrak{E}(X_A)$ and $E_{*A}$ have in a general way the character of "projective limits" of the given $E(X_B)$ and $E_{*B}$, the conditions (5) and (6) being "consistency conditions" essential to the constructive process.

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## *ERRATUM*

The eleventh (last) pentad of the second line of the value for $340!/10^{83}$ on p. 409 (August) of volume 34 of these PROCEEDINGS should read 85229 in place of 58229.

HORACE S. UHLER